

# The entangling and disentangling power of unitary transformations are unequal

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We consider two capacity quantities associated with bipartite unitary gates: the entangling and the disentangling power. For two-qubit unitaries these two capacities are always the same. Here we prove that these capacities are different in general. We do so by constructing an explicit example of a qubit-qutrit unitary whose entangling power is maximal (2 ebits), but whose disentangling power is strictly less. A corollary is that there can be no unique ordering for unitary gates in terms of their ability to perform non-local tasks. Finally we show that in large dimensions, almost all bipartite unitaries have entangling and disentangling capacities close to the maximal possible (and thus in high dimensions the difference in these capacities is small for almost all unitaries).

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**Introduction.** In quantum information theory, we wish to understand and compare states, channels or interactions via their usefulness at certain operational tasks: creation of EPR states, communication of classical or quantum bits, etc. Bipartite unitaries have proved to be a fruitful arena in which to study interactive communication tasks [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. In particular, the degree of interaction of a bipartite unitary may be quantified in a number of ways: for example its ability to perform forward or backward communication, its ability to simulate other interactions, or, as is of interest to us here, its ability to increase or decrease the entanglement between two parties.

Formally, we consider a unitary transformation  $U$  acting on a bipartite system shared by two observers Alice and Bob. Alice has a system Hilbert space  $\mathcal{H}_A$  and Bob a system Hilbert space  $\mathcal{H}_B$ . The unitary  $U = U_{AB}$  acts on  $\mathcal{H}_A \otimes \mathcal{H}_B$ . Alice (resp. Bob) also has an ancilla with Hilbert space  $\mathcal{H}_a$  (resp.  $\mathcal{H}_b$ ). Thus we extend the action of  $U$  to the full Hilbert space as  $I_a \otimes U_{AB} \otimes I_b$  where  $I_a$  (resp.  $I_b$ ) is the identity operator on  $\mathcal{H}_a$  (resp.  $\mathcal{H}_b$ ). We consider an initial state  $|\Psi^{\text{in}}\rangle$  on the full Hilbert space, then act with  $I_a \otimes U_{AB} \otimes I_b$  to produce a final state

$$|\Psi^{\text{out}}\rangle = I_a \otimes U_{AB} \otimes I_b |\Psi^{\text{in}}\rangle. \quad (1)$$

Let  $E(\Psi^{\text{in}})$  be the entanglement of  $|\Psi^{\text{in}}\rangle$ , measured by the entropy of its reduced state on the space  $\mathcal{H}_A \otimes \mathcal{H}_a$  [12]. Then the entangling power of  $U$ , which we denote  $E^\uparrow(U)$  is defined to be the maximum possible increase in the entanglement as the input state varies:

$$E^\uparrow(U) = \sup_{|\Psi^{\text{in}}\rangle} (E(\Psi^{\text{out}}) - E(\Psi^{\text{in}})). \quad (2)$$

The disentangling power of  $U$ , denoted  $E^\downarrow(U)$  is the maximum decrease in entanglement that  $U$  can effect:

$$E^\downarrow(U) = \sup_{|\Psi^{\text{in}}\rangle} (E(\Psi^{\text{in}}) - E(\Psi^{\text{out}})). \quad (3)$$

Clearly,  $E^\downarrow(U) = E^\uparrow(U^\dagger)$ . Note that by the results of [9, 10],  $E^\uparrow(U)$  is equal to the asymptotic (many copies of  $U$ ) capacity of  $U$  to generate entanglement (measured by the rate of EPR states) – in these papers it is shown that this capacity is given by the single-letter formula eq. (2), and that it is sufficient to do the optimisation over pure states.

In this paper, we prove that in general  $E^\uparrow(U)$  and  $E^\downarrow(U)$  are not equal. We show this by constructing and analyzing an explicit example in  $2 \times 3$  dimensions. It is worth recalling [8], for  $2 \times 2$ -unitaries, that  $E^\uparrow(U) = E^\downarrow(U) = E^\uparrow(U^\dagger)$ ; thus our example occurs in the smallest possible dimension.

It will be noticed that we have not said anything up to this point about the relative dimensions of the system and ancilla Hilbert spaces. It is known that for typical unitaries  $U$  it is essential to have ancillas in order to generate the maximum possible entanglement using  $U$ . A well-known extreme case is the SWAP operation on two qubits: it generates no entanglement increase if Alice and Bob each only have the qubit on which the SWAP acts, but it generates two ebits, the maximum possible increase for any unitary acting on two qubits, if Alice and Bob each have a qubit ancilla (i.e. Alice and Bob's local Hilbert space are each two qubits).

For an arbitrary  $U$  it is not known what size the ancillas need to be to reach the maximum possible entanglement increase (or decrease) for that unitary, or if indeed a maximiser exists in finite dimension. Until now this has been a major stumbling block in the calculation of the non-local capacities of interactions [9, 10].

**Two-qubit gates [8].** It is well-known that a unitary of two qubits can, up to local unitary equivalence, be written [5] as

$$U = \exp(i\alpha\sigma_x \otimes \sigma_x + i\beta\sigma_y \otimes \sigma_y + i\gamma\sigma_z \otimes \sigma_z), \quad (4)$$

with real numbers  $\alpha$ ,  $\beta$  and  $\gamma$ . Consider any input state  $|\Psi^{\text{in}}\rangle$  and output state

$$|\Psi^{\text{out}}\rangle = \exp(i\alpha\sigma_x \otimes \sigma_x + i\beta\sigma_y \otimes \sigma_y + i\gamma\sigma_z \otimes \sigma_z) |\Psi^{\text{in}}\rangle.$$

Then, taking the unitary to the other side and considering the complex conjugate, we get

$$|\overline{\Psi^{\text{in}}}\rangle = \exp(i\alpha\sigma_x \otimes \sigma_x + i\beta\sigma_y \otimes \sigma_y + i\gamma\sigma_z \otimes \sigma_z) |\overline{\Psi^{\text{out}}}\rangle,$$

because the  $\sigma_j \otimes \sigma_j$  are all real symmetric. Since for all states,  $E(\Psi) = E(\overline{\Psi})$ , we can match any entanglement increase (for input state  $|\Psi^{\text{in}}\rangle$ ) by an equal entanglement decrease (for input state  $|\overline{\Psi^{\text{out}}}\rangle$ ).

**Main result.** Our proof that the entangling and disentangling power are unequal proceeds in two steps. First we show that if a unitary transformation has the largest possible entangling power for a unitary of that dimension, then the local ancillas need only be as large as the local system Hilbert spaces (Lemma 1 below). Then we exhibit an explicit unitary acting on  $\mathbb{C}^2 \otimes \mathbb{C}^3$  which has the property that its entangling power is maximal (i.e. 2 ebits) but its disentangling power is strictly less than 2 ebits, which we prove by contradiction using our Lemma 1.

**Lemma 1** *Let  $U$  be a unitary acting on  $\mathbb{C}^A \otimes \mathbb{C}^B$  (without loss of generality we assume that  $A \leq B$ ). If  $U$  is maximally entangling (i.e.,  $E^\dagger(U) = 2 \log A$  [13]), then in eqs. (1) and (2) one may restrict to ancillas of dimension  $a = A$  and  $b = B$ ; in particular, the supremum is a maximum, achieved using an input state of the product form*

$$|\Psi^{\text{in}}\rangle_{aABb} = |\Phi\rangle_{aA} \otimes |\Psi\rangle_{Bb},$$

with  $|\Phi\rangle_{aA} = \frac{1}{\sqrt{A}} \sum_{j=1}^A |j\rangle_a |j\rangle_A$  a maximally entangled state on  $a \times A$  and some  $|\Psi\rangle_{Bb}$  on  $B \times b$ .

*Proof.* First, assume that for some ancillas of size  $a$  and  $b$ , respectively, there is actually a maximizer  $|\Psi^{\text{in}}\rangle$  – after this we will give a proof that avoids this unwarranted assumption. Generally, subadditivity of entropy [14] implies the entanglement of the final state  $E(\Psi^{\text{out}})$  satisfies

$$E(\Psi^{\text{out}}) = S(\rho_{Aa}^{\text{out}}) \leq S(\rho_a) + \log A. \quad (5)$$

Also, the triangle inequality [14] implies that

$$E(\Psi^{\text{in}}) = S(\rho_{Aa}^{\text{in}}) \geq S(\rho_a) - \log A. \quad (6)$$

(Notice that since the unitary  $U$  does not act on the ancilla Hilbert space,  $S(\rho_a)$  is the same before and after the action of  $U$ .) Thus,

$$E(\Psi^{\text{out}}) - E(\Psi^{\text{in}}) \leq 2 \log A, \quad (7)$$

but since we assumed that  $E(\Psi^{\text{out}}) - E(\Psi^{\text{in}}) = 2 \log A$ , we must have equality in eqs. (5) and (6).

Now we can calculate, using the above and the purity of the state of four parties,

$$\begin{aligned} S(\rho_{ABb}^{\text{in}}) &= S(\rho_a) \\ &= S(\rho_{Aa}^{\text{in}}) + \log A \\ &= S(\rho_{Bb}^{\text{in}}) + \log A. \end{aligned}$$

Thus we must have

$$\rho_{ABb}^{\text{in}} = \frac{1}{A} I_A \otimes \rho_{Bb}^{\text{in}}, \quad (8)$$

and we may purify the state  $\rho_{ABb}^{\text{in}}$  by writing Alice's ancilla Hilbert space in the form  $\mathcal{H}_{a_1} \otimes \mathcal{H}_{a_2}$ , so that the state of the full system is

$$|\Psi^{\text{in}}\rangle_{aABb} = |\Psi_1^{\text{in}}\rangle_{a_1A} \otimes |\Psi_2^{\text{in}}\rangle_{a_2Bb}. \quad (9)$$

We may take  $a_1$  to have dimension  $A$  and  $|\Psi_1^{\text{in}}\rangle_{Aa_1}$  is maximally entangled, and hence  $\rho_{a_1} = \frac{1}{A} I_{a_1}$ .

We now consider the state after the action of  $U$ . Eq. (5) with equality means that  $\rho_{Aa}^{\text{out}} = \frac{1}{A} I_A \otimes \rho_a$ , so that

$$\begin{aligned} \rho_{Aa_1a_2}^{\text{out}} &= \frac{1}{A} I_A \otimes \rho_a \\ &= \frac{1}{A} I_A \otimes \frac{1}{A} I_{a_1} \otimes \rho_{a_2}. \end{aligned} \quad (10)$$

Hence,

$$\begin{aligned} E(\Psi^{\text{in}}) &= S(\rho_{a_2}), \\ E(\Psi^{\text{out}}) &= S(\rho_{a_2}) + 2 \log A, \end{aligned}$$

from eqs. (9) and (10).

We may now see that there is a different initial state which yields the same entanglement increase. We take exactly the state (9) but now consider the situation in which the ancilla particle  $a_2$  is transferred to Bob — let us relabel  $\tilde{a} = a_1$  and  $\tilde{b} = ba_2$ . Thus consider the initial state

$$|\tilde{\Psi}^{\text{in}}\rangle_{\tilde{a}AB\tilde{b}} = |\Psi_1^{\text{in}}\rangle_{\tilde{a}A} \otimes |\Psi_2^{\text{in}}\rangle_{B\tilde{b}}. \quad (11)$$

This state has

$$\begin{aligned} E(\tilde{\Psi}^{\text{in}}) &= 0, \\ E(\tilde{\Psi}^{\text{out}}) &= 2 \log A, \end{aligned}$$

and we are done.

However, as we have said at the beginning of this proof, we cannot assume that the supremum in eq. (2) is a maximum. Instead, for every  $\epsilon > 0$  there exist ancilla dimensions  $a$  and  $b$  and an initial state such that

$$2 \log A - \epsilon \leq E(\Psi^{\text{out}}) - E(\Psi^{\text{in}}) \leq 2 \log A. \quad (12)$$

For eqs. (5) and (6) this implies

$$S(\rho_a) + \log A - \epsilon \leq E(\Psi^{\text{out}}) \leq S(\rho_a) + \log A, \quad (13)$$

$$S(\rho_a) - \log A \leq E(\Psi^{\text{in}}) \leq S(\rho_a) - \log A + \epsilon. \quad (14)$$

Eq. (14) gives that for the mutual information of the initial state between  $A$  and  $Bb$ ,

$$\begin{aligned} I(A : Bb)_{\Psi^{\text{in}}} &= S(\rho_A^{\text{in}}) + S(\rho_{Bb}^{\text{in}}) - S(\rho_{ABb}^{\text{in}}) \\ &= S(\rho_A^{\text{in}}) + E(\Psi^{\text{in}}) - S(\rho_a) \\ &\leq \log A + S(\rho_a) - \log A + \epsilon - S(\rho_a) = \epsilon. \end{aligned}$$

Since the mutual information can be expressed by means of the relative entropy,

$$I(A : Bb)_{\Psi^{\text{in}}} = D(\rho_{ABb}^{\text{in}} \| \rho_A^{\text{in}} \otimes \rho_{Bb}^{\text{in}}),$$

and with Pinsker's inequality,

$$D(\rho \| \sigma) \geq \left( \frac{1}{2} \|\rho - \sigma\|_1 \right)^2,$$

we find that

$$\|\rho_{ABb}^{\text{in}} - \rho_A^{\text{in}} \otimes \rho_{Bb}^{\text{in}}\|_1 \leq 2\sqrt{\epsilon}. \quad (15)$$

The second state is a product state, so it has a purification of product form,  $|\hat{\Psi}_1^{\text{in}}\rangle_{a_1 A} \otimes |\hat{\Psi}_2^{\text{in}}\rangle_{a_2 Bb}$ , and by Uhlmann's theorem [15] one can isometrically map  $a$  to  $a_1 a_2$  such that

$$\left\| |\Psi^{\text{in}}\rangle_{a_1 a_2 ABb} - |\hat{\Psi}_1^{\text{in}}\rangle_{a_1 A} \otimes |\hat{\Psi}_2^{\text{in}}\rangle_{a_2 Bb} \right\|_1 \leq 2\sqrt[4]{\epsilon} =: \delta, \quad (16)$$

where we have used a well-known relation between fidelity and trace distance [16]. Now we will switch over to

$$|\hat{\Psi}^{\text{in}}\rangle_{a_1 a_2 ABb} = |\hat{\Psi}_1^{\text{in}}\rangle_{a_1 A} \otimes |\hat{\Psi}_2^{\text{in}}\rangle_{a_2 Bb}$$

as the new input state. Notice that, because  $a$  is not affected by the dynamics,

$$\begin{aligned} E(\Psi^{\text{out}}) - E(\Psi^{\text{in}}) &= S(Aa)_{\Psi^{\text{out}}} - S(Aa)_{\Psi^{\text{in}}} \\ &= S(A|a)_{\Psi^{\text{out}}} - S(A|a)_{\Psi^{\text{in}}}, \end{aligned} \quad (17)$$

and likewise for the new state  $\hat{\Psi}$ :

$$\begin{aligned} E(\hat{\Psi}^{\text{out}}) - E(\hat{\Psi}^{\text{in}}) &= S(Aa)_{\hat{\Psi}^{\text{out}}} - S(Aa)_{\hat{\Psi}^{\text{in}}} \\ &= S(A|a)_{\hat{\Psi}^{\text{out}}} - S(A|a)_{\hat{\Psi}^{\text{in}}}, \end{aligned} \quad (18)$$

with the conditional entropy [17]  $S(X|Y)_\rho = S(\rho_{XY}) - S(\rho_Y)$ . This means we have only to control how much the conditional entropy changes when we modify the state, and this we can indeed do with the help of a variant of Fannes' inequality, proved recently [18]: for  $\delta \leq 1/2$ ,

$$\left| S(A|a)_{\hat{\Psi}^{\text{in}}} - S(A|a)_{\Psi^{\text{in}}} \right| \leq 2H_2(\delta) + 4\delta \log A,$$

and likewise for the output states, where we have used the binary entropy  $H_2(\delta) = -\delta \log \delta - (1-\delta) \log(1-\delta)$ . (Note that, unlike the usual Fannes inequality, we have only a dependence on the dimension of  $A$  but not of the ancilla.)

But now we can perform the same trick as above: we look at the new input state obtained from  $|\hat{\Psi}^{\text{in}}\rangle_{a_1 a_2 ABb}$  by handing  $a_2$  to Bob, i.e., with  $\tilde{a} = a_1$  and  $\tilde{b} = ba_2$ ,

$$|\hat{\Psi}^{\text{in}}\rangle_{\tilde{a} A \tilde{b}} = |\hat{\Psi}_1^{\text{in}}\rangle_{a_1 A} \otimes |\hat{\Psi}_2^{\text{in}}\rangle_{Bba_2}.$$

Since both input and output state, restricted to  $a_1 a_2 A$ , are then products across  $a_1 A - a_2$ , we get

$$\begin{aligned} E(\tilde{\Psi}^{\text{out}}) - E(\tilde{\Psi}^{\text{in}}) &= S(\tilde{\rho}_{Aa_1}^{\text{out}}) - S(\tilde{\rho}_{Aa_1}^{\text{in}}) \\ &= S(\tilde{\rho}_{Aa_1 a_2}^{\text{out}}) - S(\tilde{\rho}_{Aa_1 a_2}^{\text{in}}) \\ &= E(\hat{\Psi}^{\text{out}}) - E(\hat{\Psi}^{\text{in}}) \\ &\geq 2 \log A - \epsilon - 4H_2(\delta) - 8\delta \log A. \end{aligned}$$

At this point we can perform the limit  $\epsilon$  (and hence  $\delta$ )  $\rightarrow 0$ : since  $\tilde{a}$  purifies  $A$  and  $\tilde{b}$  purifies  $B$ , we can restrict their dimensions to  $A$  and  $B$ , respectively, and so the states  $|\tilde{\Psi}^{\text{in}}\rangle$  have an accumulation point for which the difference between the output and the input entanglement is precisely  $2 \log A$ . This state is a product state between Alice and Bob, and just as in the earlier argument, it is now immediate that Alice must have a maximally entangled state between  $A$  and her ancilla.  $\square$

We thus conclude that if a unitary creates the maximal amount of entanglement, it can do so by acting on a state which is product pure state between Alice and Bob. Furthermore the state on Alice's side may be taken to be maximally entangled between the system and ancilla. If  $\mathcal{H}_A$  and  $\mathcal{H}_B$  have different dimensions (with  $\mathcal{H}_A$  assumed to be smaller) we may only conclude that Bob's initial state may be taken to be pure with an ancilla of the same size as  $\mathcal{H}_B$ .

Of course if the dimensions of  $\mathcal{H}_A$  and  $\mathcal{H}_B$  are equal then we may run the argument again, with the roles of Alice and Bob interchanged, to show that the initial state may be taken to be a product state between Alice and Bob, with both Alice and Bob maximally entangled with their local ancilla. From this it is not hard to show that in this case, still assuming that  $U$  is maximally entangling, it is also maximally disentangling. In other words, for  $A = B$ ,

$$E^\uparrow(U) = 2 \log A \iff E^\downarrow(U) = 2 \log A.$$

We now exhibit an explicit unitary transformation acting on  $\mathbb{C}^2 \otimes \mathbb{C}^3$  which can entangle maximally, but which cannot disentangle maximally: consider

$$\begin{aligned} U_{2 \times 3} &= -i |w_{00}\rangle \langle 00| + |w_{01}\rangle \langle 01| + |w_{02}\rangle \langle 02| \\ &\quad + |w_{10}\rangle \langle 10| + |w_{11}\rangle \langle 11| - i |w_{12}\rangle \langle 12|, \end{aligned} \quad (19)$$

with (for  $j = 0, 1, 2$ )

$$\begin{aligned} |w_{0j}\rangle &= \frac{1}{\sqrt{3}} (|\alpha\rangle |0\rangle + \omega^j |\beta\rangle |1\rangle + \omega^{2j} |\gamma\rangle |2\rangle), \\ |w_{1j}\rangle &= \frac{1}{\sqrt{3}} (|\alpha^\perp\rangle |0\rangle + \omega^j |\beta^\perp\rangle |1\rangle + \omega^{2j} |\gamma^\perp\rangle |2\rangle). \end{aligned}$$

Here  $\omega = e^{2\pi i/3}$  is the cube-root of unity, and  $|\alpha\rangle, |\beta\rangle,$

$|\gamma\rangle$  and  $|\alpha^\perp\rangle, |\beta^\perp\rangle, |\gamma^\perp\rangle$  are sets of “trine” states:

$$\begin{aligned} |\alpha\rangle &= |0\rangle, & |\alpha^\perp\rangle &= |1\rangle, \\ |\beta\rangle &= -\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle, & |\beta^\perp\rangle &= -\frac{1}{2}|1\rangle - \frac{\sqrt{3}}{2}|0\rangle, \\ |\gamma\rangle &= -\frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle, & |\gamma^\perp\rangle &= -\frac{1}{2}|1\rangle + \frac{\sqrt{3}}{2}|0\rangle. \end{aligned}$$

The unitary  $U_{2\times 3}$  can create two ebits. For consider its action on the state

$$|\Phi_1^{\text{in}}\rangle = \frac{1}{2}(|0\rangle_a|0\rangle_A + |1\rangle_a|1\rangle_A) \otimes (|0\rangle_B|0\rangle_b + |2\rangle_B|2\rangle_b).$$

The subscript  $A$  denotes Alice’s system state and  $a$  her ancilla state; similarly  $B$  denotes Bob’s system state and  $b$  his ancilla state. The unitary  $U_{2\times 3}$  acts on the Hilbert spaces  $A$  and  $B$  (i.e. the full unitary is  $I_a \otimes U_{2\times 3} \otimes I_b$ , where  $I_a$  is the identity operator on the  $a$  Hilbert space). Clearly the initial state  $|\Phi_1^{\text{in}}\rangle$  has zero entanglement between Alice and Bob (i.e. between  $Aa$  and  $Bb$ ). It is not difficult to check that the final state

$$|\Phi_1^{\text{out}}\rangle = U_{2\times 3} |\Phi_1^{\text{in}}\rangle \quad (20)$$

has entanglement of two ebits between Alice and Bob. Thus  $U_{2\times 3}$  has the maximum possible entangling power for any unitary on  $\mathbb{C}^2 \otimes \mathbb{C}^3$ .

We now show that the disentangling power of  $U_{2\times 3}$  is strictly less than 2 ebits. It will be convenient to analyze the entangling power of the inverse of  $U_{2\times 3}$ . Assuming the contrary, by Lemma 1, if  $U_{2\times 3}$  would entangle maximally, the state from which it creates most entanglement may be taken to be a product state:  $|\eta_1\rangle_{aA} \otimes |\eta_2\rangle_{Bb}$ , and without loss of generality  $|\eta_1\rangle_{aA}$  is a maximally entangled state of two qubits,  $|\eta_2\rangle_{Bb}$  an arbitrary pure state of two qutrits. Thus the most general input state we need to consider is

$$\begin{aligned} |\Phi_2^{\text{in}}\rangle &= \frac{1}{\sqrt{2}}(|0\rangle_a|0\rangle_A + |1\rangle_a|1\rangle_A) \\ &\otimes (|0\rangle_B|\tau_0\rangle_b + |1\rangle_B|\tau_1\rangle_b + |2\rangle_B|\tau_2\rangle_b). \end{aligned} \quad (21)$$

Normalization of  $|\Phi_2^{\text{in}}\rangle$  means that

$$\langle\tau_0|\tau_0\rangle_b + \langle\tau_1|\tau_1\rangle_b + \langle\tau_2|\tau_2\rangle_b = 1. \quad (22)$$

Clearly  $|\Phi_2^{\text{in}}\rangle$  has no entanglement between  $Aa$  and  $Bb$ .

To compute the output state, we begin by rewriting

the inverse  $U_{2\times 3}^\dagger$  as

$$\begin{aligned} U_{2\times 3}^\dagger &= |0\rangle_A|v_0\rangle_B \langle 0|_A \langle 0|_B \\ &+ \left( -\frac{1}{2}|0\rangle_A|v_1\rangle_B - \frac{\sqrt{3}}{2}|1\rangle_A|v'_1\rangle_B \right) \langle 0|_A \langle 1|_B \\ &+ \left( -\frac{1}{2}|0\rangle_A|v_2\rangle_B + \frac{\sqrt{3}}{2}|1\rangle_A|v'_2\rangle_B \right) \langle 0|_A \langle 2|_B \\ &+ |1\rangle_A|v'_0\rangle_B \langle 1|_A \langle 0|_B \\ &+ \left( \frac{\sqrt{3}}{2}|0\rangle_A|v_1\rangle_B - \frac{1}{2}|1\rangle_A|v'_1\rangle_B \right) \langle 1|_A \langle 1|_B \\ &+ \left( -\frac{\sqrt{3}}{2}|0\rangle_A|v_2\rangle_B - \frac{1}{2}|1\rangle_A|v'_2\rangle_B \right) \langle 1|_A \langle 2|_B, \end{aligned}$$

where (for  $j = 0, 1, 2$ )

$$\begin{aligned} |v_j\rangle &= \frac{1}{\sqrt{3}}(i|0\rangle + \omega^{-j}|1\rangle + \omega^{-2j}|2\rangle), \\ |v'_j\rangle &= \frac{1}{\sqrt{3}}(|0\rangle + \omega^{-j}|1\rangle + i\omega^{-2j}|2\rangle). \end{aligned}$$

Thus the result of  $U_{2\times 3}^\dagger$  acting on (21) is

$$\begin{aligned} |\Phi_2^{\text{out}}\rangle &= U_{2\times 3}^\dagger |\Phi_2^{\text{in}}\rangle \\ &= \frac{1}{2} \left[ |0\rangle_a|0\rangle_A |\Phi_{00}\rangle_{Bb} + |0\rangle_a|1\rangle_A |\Phi_{01}\rangle_{Bb} \right. \\ &\quad \left. + |1\rangle_a|0\rangle_A |\Phi_{10}\rangle_{Bb} + |1\rangle_a|1\rangle_A |\Phi_{11}\rangle_{Bb} \right], \end{aligned}$$

where now

$$\begin{aligned} |\Phi_{00}\rangle &= \sqrt{2} \left[ |v_0\rangle_B |\tau_0\rangle_b - \frac{1}{2}|v_1\rangle_B |\tau_1\rangle_b - \frac{1}{2}|v_2\rangle_B |\tau_2\rangle_b \right], \\ |\Phi_{01}\rangle &= \sqrt{2} \left[ -\frac{\sqrt{3}}{2}|v'_1\rangle_B |\tau_1\rangle_b + \frac{\sqrt{3}}{2}|v'_2\rangle_B |\tau_2\rangle_b \right], \\ |\Phi_{10}\rangle &= \sqrt{2} \left[ \frac{\sqrt{3}}{2}|v_1\rangle_B |\tau_1\rangle_b - \frac{\sqrt{3}}{2}|v_2\rangle_B |\tau_2\rangle_b \right], \\ |\Phi_{11}\rangle &= \sqrt{2} \left[ |v'_0\rangle_B |\tau_0\rangle_b - \frac{1}{2}|v'_1\rangle_B |\tau_1\rangle_b - \frac{1}{2}|v'_2\rangle_B |\tau_2\rangle_b \right]. \end{aligned}$$

Now, in order for  $|\Phi_2^{\text{out}}\rangle$  to be maximally entangled we require that the four states  $|\Phi_{00}\rangle, |\Phi_{01}\rangle, |\Phi_{10}\rangle$  and  $|\Phi_{11}\rangle$  form an orthonormal basis:

$$\langle\Phi_{ij}|\Phi_{km}\rangle = \delta_{ij,km}.$$

This puts constraints on the  $|\tau_j\rangle$ , which, as we shall see, lead to a contradiction.

Bearing in mind the normalization of the  $|\tau_j\rangle$ , eq. (22), the four equations expressing the condition that the vectors  $|\Phi_{ij}\rangle$  be normalised are all the same, namely:

$$\langle\tau_1|\tau_1\rangle + \langle\tau_2|\tau_2\rangle = \frac{2}{3}, \text{ or equivalently, } \langle\tau_0|\tau_0\rangle = \frac{1}{3}.$$

The requirement that  $\langle\Phi_{00}|\Phi_{10}\rangle = 0$  thus leads to

$$\langle\tau_0|\tau_0\rangle = \langle\tau_1|\tau_1\rangle = \langle\tau_2|\tau_2\rangle = \frac{1}{3}. \quad (23)$$

The requirement that  $\langle \Phi_{01} | \Phi_{10} \rangle = 0$  yields

$$-\langle \tau_1 | \tau_1 \rangle - \langle \tau_2 | \tau_2 \rangle + (1 - \sqrt{3})\omega^2 \langle \tau_1 | \tau_2 \rangle + (1 + \sqrt{3})\omega \langle \tau_2 | \tau_1 \rangle = 0.$$

This has the unique solution  $\langle \tau_1 | \tau_2 \rangle = \frac{\omega}{3}$ , and with Cauchy-Schwarz and eq. (23), this means that

$$|\tau_2\rangle = \omega |\tau_1\rangle. \quad (24)$$

The requirement that  $\langle \Phi_{00} | \Phi_{01} \rangle = 0$  gives

$$\begin{aligned} & -(1 + \sqrt{3})\omega^2 \langle \tau_0 | \tau_1 \rangle + (1 - \sqrt{3})\omega \langle \tau_0 | \tau_2 \rangle \\ & + \frac{1}{2} \langle \tau_1 | \tau_1 \rangle - \frac{1}{2} (1 + \sqrt{3})\omega^2 \langle \tau_1 | \tau_2 \rangle \\ & + \frac{1}{2} (1 - \sqrt{3})\omega \langle \tau_2 | \tau_1 \rangle - \frac{1}{2} \langle \tau_2 | \tau_2 \rangle = 0 \end{aligned}$$

Using eqs. (23) and (24), this implies that

$$\langle \tau_0 | \tau_1 \rangle = -\frac{\omega}{6} \quad \text{and} \quad \langle \tau_0 | \tau_2 \rangle = -\frac{\omega^2}{6}. \quad (25)$$

But now, inserting eqs. (23), (24) and (25), we get

$$\begin{aligned} \langle \Phi_{00} | \Phi_{11} \rangle &= \frac{2}{3} \langle \tau_0 | \tau_0 \rangle - \frac{1}{3} (1 + \sqrt{3})\omega^2 \langle \tau_0 | \tau_1 \rangle \\ &\quad - \frac{1}{3} (1 - \sqrt{3})\omega \langle \tau_0 | \tau_2 \rangle - \frac{1}{3} (1 - \sqrt{3})\omega \langle \tau_1 | \tau_0 \rangle \\ &\quad + \frac{1}{6} \langle \tau_1 | \tau_1 \rangle + \frac{1}{6} (1 + \sqrt{3})\omega^2 \langle \tau_1 | \tau_2 \rangle + \frac{1}{6} \langle \tau_2 | \tau_2 \rangle \\ &\quad - \frac{1}{3} (1 + \sqrt{3})\omega^2 \langle \tau_2 | \tau_0 \rangle + \frac{1}{6} (1 - \sqrt{3})\omega \langle \tau_2 | \tau_1 \rangle \\ &= \frac{2}{3} \neq 0. \end{aligned}$$

Thus there is no choice of  $|\tau_0\rangle, |\tau_1\rangle, |\tau_2\rangle$  for  $|\Phi_{ij}\rangle$  to form an orthonormal basis. This is the desired contradiction, and we conclude that  $E^\downarrow(U) < 2 = E^\uparrow(U)$ .

**Further thoughts and conclusion.** We have found an example of bipartite unitary of smallest possible dimension such that its entangling and its disentangling power are different. This is a striking result as it shows that there can be no unique ordering of unitary gates with respect to their various capacities. For consider  $U_1 = U_{2 \times 3}$  and  $U_2 = U_{2 \times 3}^\dagger$ :  $U_1$  has greater entangling capacity than  $U_2$ ; but  $U_1$  has smaller disentangling capacity than  $U_2$ . Note however, that our proof is only by contradiction, and that in particular we show only that there is a difference between  $E^\uparrow$  and  $E^\downarrow$  but not how large it is.

We have done some numerical work, which, for our gate  $U_{2 \times 3}$ , indicates that

$$2 - E^\downarrow(U_{2 \times 3}) \approx 0.06.$$

Furthermore, we tried to find the maximum difference  $E^\uparrow(U) - E^\downarrow(U)$  over all  $2 \times 3$ -gates  $U$ , which seems to

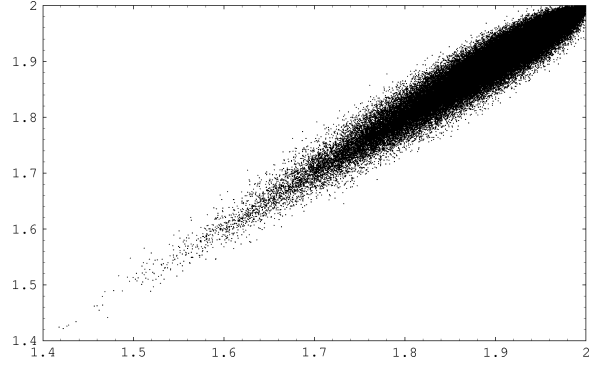


FIG. 1: Entangling vs. disentangling power for 170,000 randomly chosen  $2 \times 3$  unitaries. The local ancilla dimensions are 2 and 3 respectively.

be  $\approx 0.13$ , and in general for randomly selected unitary, the entangling and the disentangling power doesn't seem to be much different. [See Fig 1.]

We can explain this partly by the *concentration of measure phenomenon* in large dimensions [which usually however kicks in for relatively small dimensions]: For a (random, according to Haar measure) unitary  $U_{AB}$  in  $A \times B$  dimensions, consider as initial state

$$|\Phi^{\text{in}}\rangle = |\Phi_A\rangle_{aA} \otimes |\Phi_B\rangle_{Bb},$$

the tensor product of two maximally entangled states between the local systems and the respective local ancillas. As before, we will assume without loss of generality that  $A \leq B$ . On the face of it, it might be thought that this state may not be the best input state for a particular unitary, enabling us to achieve the entangling capacity; and indeed it will not be for every unitary. However, in fact, it is a pretty good input state for most unitaries. For we shall show that the expected entanglement of  $|\Phi^{\text{out}}\rangle$  for this state is close to  $2 \log A$ ; to be precise,

$$\mathbb{E}_U E(\Phi^{\text{out}}) \geq 2 \log A - \frac{1}{\ln 2} \frac{A^2}{B^2}. \quad (26)$$

For this, we use the inequality  $S(\rho) = -\text{Tr} \rho \log \rho \geq -\log \text{Tr} \rho^2 = S_2(\rho)$  between the von Neumann and the Rényi entropy, and then have

$$\begin{aligned} \mathbb{E}_U E(\Phi^{\text{out}}) &\geq \mathbb{E}_U \left[ -\log \text{Tr}(\rho_{aA}^{\text{out}})^2 \right] \\ &\geq -\log \mathbb{E}_U \text{Tr}(\rho_{aA}^{\text{out}})^2, \end{aligned}$$

by the convexity of  $-\log x$ . To evaluate the quadratic average in the last line, we rewrite it as follows:

$$\begin{aligned} \text{Tr}(\rho_{aA}^{\text{out}})^2 &= \text{Tr} \left( (\rho_{aA}^{\text{out}} \otimes \rho_{aA}^{\text{out}}) F_{aA,aA} \right) \\ &= \text{Tr} \left( (\Phi_{aABb}^{\text{out}} \otimes \Phi_{aABb}^{\text{out}}) (F_{aA,aA} \otimes I_{BbBb}) \right), \end{aligned}$$

with the SWAP operator  $F$  on  $aAaA$ . Hence

$$\mathbb{E}_U \text{Tr}(\rho_{aA}^{\text{out}})^2 = \text{Tr} \left( \omega (F_{aA,aA} \otimes I_{BbBb}) \right),$$

where — with the symmetric and antisymmetric states

$$\sigma_{AB} = \frac{I_{ABAB} + F_{AB,AB}}{AB(AB+1)} = \frac{2}{AB(AB+1)} \Pi_{\text{sym}} \text{ and}$$

$$\alpha_{AB} = \frac{I_{ABAB} - F_{AB,AB}}{AB(AB-1)} = \frac{2}{AB(AB-1)} \Pi_{\text{anti}},$$

respectively —

$$\begin{aligned} \omega &= (\text{id}_{abb} \otimes \mathcal{T}_{AB,AB})(\Phi^{\text{in}} \otimes \Phi^{\text{in}}) \\ &= \frac{1}{2} \left(1 + \frac{1}{AB}\right) \sigma_{ab} \otimes \sigma_{AB} + \frac{1}{2} \left(1 - \frac{1}{AB}\right) \alpha_{ab} \otimes \alpha_{AB}. \end{aligned}$$

Here,

$$\begin{aligned} \mathcal{T}_{AB,AB}(\rho) &= \int dU (U \otimes U) \rho (U \otimes U)^\dagger \\ &= \sigma_{AB} \text{Tr}(\rho \Pi_{\text{sym}}) + \alpha_{AB} \text{Tr}(\rho \Pi_{\text{anti}}) \end{aligned}$$

is the twirling operation [19] — the second line follows from the fact that the  $U \otimes U$ -representation decomposes into the symmetric and the antisymmetric part. Using the formulas  $\text{Tr}_{BB} I_{ABAB} = B^2 I_{AA}$  and  $\text{Tr}_{BB} F_{AB,AB} = B F_{A,A}$ , this yields straightforwardly

$$\mathbb{E}_U \text{Tr}(\rho_{aA}^{\text{out}})^2 = \frac{A^2 + B^2 - 2}{A^2 B^2 - 1} \leq \frac{1}{A^2} + \frac{1}{B^2},$$

which immediately implies eq. (26). A deep result from probability theory, *Levy's lemma* (see [20]), as applied in [21], now informs us that the probability of a random  $U$  yielding less than  $2 \log A - \frac{1}{\ln 2} \frac{A^2}{B^2} - \epsilon$  ebits for  $|\Phi^{\text{out}}\rangle$  is bounded above by

$$\exp\left(-\frac{\text{const.}}{(\log A)^2} \epsilon^2 AB\right).$$

Since with  $U$  also  $U^\dagger$  is Haar distributed, we conclude that (for large  $A \leq B$  or for  $B$  much larger than  $A$ ) a random  $U$  is overwhelmingly likely to have entangling and disentangling power close to the maximum  $2 \log A$  (and thus the difference between these capacities is also likely to be small).

We note however, that this does not preclude the possibility that a particular unitary could have entangling and disentangling power very different from each other. [Although our numerical evidence, described above, shows that this does not seem to happen in dimension  $2 \times 3$  for example]. Indeed, independent work by Harrow and Shor [22] shows that for large local dimensions  $d$ , it is possible to construct a unitary for which  $E^\dagger(U) - E^1(U) \sim \log d$ .

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